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Heat Flow in an Exactly Solvable Model

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A chain of one-dimensional oscillators is considered. They are mechanically uncoupled and interact via a stochastic process which redistributes the energy between nearest neighbors. The total energy is kept constant except for the interactions of the extremal oscillators with reservoirs at different temperatures. The stationary measures are obtained when the chain is finite; the thermodynamic limit is then considered, approach to the Gibbs distribution is proven, and a linear temperature profile is obtained.

KEY WORDS: Heat conduction; Fourier's law; stochastic dynamics for infinitely many particles; additive processes.

1. INTRODUCTION

The aim of this paper is to obtain rigorous results for the heat conductivity in lattice systems. We will consider a system of one-dimensional harmonic oscillators. It is well known that harmonic chains do not obey Fourier's law (see, for instance, Ref. 1); however, they have been extensively studied because somehow they are more treatable from a mathematical point of view. Nonharmonic effects rule the heat flow in more realistic models and so they should be taken into account. A way, which is not, of course, the only possible one, is to simulate them by stochastic processes whose result is to lead the chain to the "right" stationary state which describes the heat flow. For instance, in Ref. 2 each oscillator was coupled to a reservoir at a definite temperature (different for each oscillator) via a Fokker–Planck force. The temperature profile is chosen so that these reservoirs on the

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average do not give energy to the chain and therefore do not contribute to the heat flux.

In Ref. 6 a completely mechanical system is considered. This is a one-dimensional quantum model. Finitely many atoms interact pairwise via a common coupling with suitable intermediate reservoirs. The ends of the chain interact with reservoirs at different temperature. The model is studied in the so-called "weak coupling limit," and it becomes a stochastic process analogous to the one we consider here and essentially the same as that treated in Ref. 3. The stationary measures exhibit a temperature profile and a heat flux which satisfies Fourier's law.

In this paper we consider uncoupled one-dimensional oscillators and we simulate the effect of the "forces" leading to the stationary state by a stochastic process which redistributes the energy between nearest-neighbor oscillators keeping constant the total energy of the system. The oscillators at the extremes of the chain are coupled to reservoirs at different temperatures according to a Glauber process and in this way energy is transferred to the system. While this looks quite rough from a physical point of view, it has, surprisingly enough, the advantage of being exactly solvable.

An analogous way to simulate local "thermalization" was introduced in Ref. 3. By this procedure measures are obtained that are locally similar to Gibbs distributions, whose temperatures are space dependent. These are not the only measures exhibiting a temperature gradient. For instance, in Ref. 4 different measures are introduced in a very natural way and other procedures are easily conceivable. A physically correct choice among these possibilities is an interesting and still open problem.

In Section 2 we introduce in a more detailed way the model we will consider and we reduce its analysis to the study of a system of interacting random particles. This is accomplished by using the techniques of "association"; for a review on this point see, for instance, Ref. 5. The probability estimates are obtained in Section 3 and the results are reported in Section 4.

2. THE MODEL AND THE ASSOCIATE PROCESS

We consider a system of one-dimensional oscillators; for $x = -L, -L+1, \ldots, L$, $q_x, p_x \in \mathbb{R} \times \mathbb{R}$ describe, respectively, position and velocity of the "x" oscillator, the phase space being therefore $(\mathbb{R} \times \mathbb{R})^{2L+1}$. The oscillators are mechanically uncoupled; the energy is the sum of the energies of the single oscillators, i.e., $\sum_x q_x^2 + p_x^2$ (in suitable units). The system undergoes a stochastic time evolution defined as follows: choose a couple of nearest-neighbor sites and let the oscillators exchange energy according to a microcanonical procedure, i.e., keep the total energy fixed and redistribute it with uniform distribution on the surface of constant energy. At the boundaries, $\pm L$, thermalize the oscillators according to the Gibbs distribution with temperatures $T_\pm$ $(T_+ \neq T_-)$. It is convenient to introduce the random variables ξ_x, $x \in [-L, L]$, taking values in $\mathbb{R}_+$; ξ_x gives the energy of the xth oscillator. It is easy to see that in the stationary distribution the conditional probability of q_x, p_x given ξ_x is the Lebesgue measure on the sphere $q_x^2 + p_x^2 = \xi_x$. Therefore we will restrict our considerations to the process for the variables ξ_x, $x \in [-L, L]$.

Lemma 2.1. The values ξ'_x, ξ'_{x+1} of the energies at $x, x+1$ when the couple $x, x+1$ is chosen are given by

$$\xi'_x = p(\xi_x + \xi_{x+1}) \qquad \xi'_{x+1} = (1-p)(\xi_x + \xi_{x+1}), \qquad p \in [0, 1] \tag{2.1}$$

where ξ_x and ξ_{x+1} are the values of the energies before the rearrangement. The distribution of p is uniform, i.e., the Lebesgue measure in $[0, 1]$.

The proof of this lemma is elementary and so it is omitted.

Let

$$\xi^{(L)} = (\xi_{-L}, \ldots, \xi_L) \in \mathbb{R}_+^{2L+1} \tag{2.2}$$

The generator $G^{(L)}$ of the stochastic evolution we described before is given by

$$\begin{aligned}(G^{(L)}f)(\xi^{(L)}) = \sum_{x=-L}^{L-1} \int_0^1 dp \big[f(\xi_{-L}, \ldots, p(\xi_x + \xi_{x+1}),& \\ (1-p)(\xi_x + \xi_{x+1}), \ldots, \xi_L) - f(\xi^{(L)}) \big]& \\ + \int_0^\infty d\xi' \, \beta_- e^{-\beta_- \xi'} \big[f(\xi', \ldots, \xi_L) - f(\xi^{(L)}) \big]& \\ + \int_0^\infty d\xi' \, \beta_+ e^{-\beta_+ \xi'} \big[f(\xi_{-L}, \ldots, \xi') - f(\xi^{(L)}) \big]&\end{aligned} \tag{2.3}$$

with f function on $\mathbb{R}_+^{2L+1}$, $\beta e^{-\beta\xi} d\xi$ the Gibbs distribution for the energy of a harmonic oscillator at temperature $T = 1/K\beta$, and β_- $[\beta_+]$ the inverse temperature of the left (right) reservoir. It is easy to see that the Markov process with generator $G^{(L)}$ has a unique stationary measure, μ_L. In the limit $L \to \infty$, μ_L should approach the Gibbs measure, namely, ξ_x should have exponential distribution. The natural quantities to compute are therefore the mean values of the functions

$$F(k, \xi) = \prod_x \frac{\xi_x^{k_x}}{k_x!}, \qquad \sum_x k_x < \infty, \qquad k = (k_x)_{x \in \mathbb{Z}} \tag{2.4}$$

because for an exponential law we have

$$\int_0^\infty \frac{x^k}{k!}\lambda e^{-\lambda x}\,dx = \lambda^{-k}$$

We now introduce a family of (continuous time) Markov processes which are related to the one generated by $G^{(L)}$ via Theorem 2.1 which will be stated below.

Definition 2.1. Given L let

$$\delta(\pm) = \pm(L+1) \tag{2.5}$$

$$I_L = [-L, L] \cup \delta(-) \cup \delta(+) \tag{2.6}$$

We then consider the Markov process on $\mathbb{N}^{I_L}$ ($\mathbb{N} = 0, 1, \ldots$] whose generator A_L is given by

$$\begin{aligned}(A_L f)&(n_{\delta(-)}, n_{-L}, \ldots, n_L, n_{\delta(+)})\\ &= \sum_{i=-L}^{L-1} \frac{1}{n_i + n_{i-1} + 1} \sum_{q=0}^{n_i+n_{i+1}} \big[f(n_{\delta(-)}, n_{-L}, \ldots, n_{i-1}, q,\\ &\qquad\qquad n_i + n_{i+1} - q, \ldots, n_L, n_{\delta(+)})\\ &\qquad\qquad - f(n_{\delta(-)}, \ldots, n_{\delta(+)})\big]\\ &\quad + f(n_{\delta(-)} + n_{-L}, 0, \ldots, n_{\delta(+)}) - f(n_{\delta(-)}, n_{-L}, \ldots, n_L, n_{\delta(+)})\\ &\quad + f(n_{\delta(-)}, \ldots, n_{L-1}, 0, n_{\delta(+)} + n_L) - f(n_{\delta(-)}, n_{-L}, \ldots, n_L, n_{\delta(+)})\end{aligned} \tag{2.7}$$

We denote by $\mathbb{Q}_n^{(L)}$ the Markov process generated by A_L, n_t will be the family of random variables $n_{\delta(-)}, \ldots, n_{\delta(+)}$ at time t, $n_0 = n$. $\mathbb{Q}_n^{(L)}$ describes the motion of $|n|$ $(= n_{\delta(-)} + \cdots + n_{\delta(+)})$ indistinguishable particles which move on I_L and are stuck when arriving at $\delta(\pm)$, absorbing boundary conditions at $\delta(\pm)$. In the interior the motion is specified by the following rule. At each pair of sites x, $x+1$, $-L \leqslant x \leqslant L-1$, there is a clock which rings with exponential law, the clocks ring independently from each other. When the x, $x+1$ clock rings the particles at x and $x+1$ redistribute among these sites. Namely, let n_x be the number of particles at x, n_{x+1} at $x+1$, then choose p uniformly between $0, 1, \ldots, n_x + n_{x+1}$ and put p particles at x and $n_x + n_{x+1} - p$ at $x+1$. There are clocks at $\delta(\pm)$ also, and they describe the absorption of particles at $\delta(\pm)$, namely, when the $\delta_{(+)}[\delta_{(-)}]$ clock rings the particles at $+L[-L]$ go to $\delta(+)[\delta(-)]$, where they will remain forever.

Notice that during this evolution the total number of particles is constant, and that those stuck at the boundaries must increase. It is clear that "eventually" all the particles get out of $[-L, L]$. Therefore the asymptotic properties of this motion are completely described by the exit distribution.

In the sequel we will use the following notation.

Definition 2.2. Let

$$k=(k_{-L},\ldots,k_L), \qquad \hat{k}=(0,k_{-L},\ldots,k_L,0)$$

$$n=(n_{\delta(-)},n_{-L},\ldots,n_L,n_{\delta(+)}), \qquad \check{n}=(n_{-L},\ldots,n_L)$$

$$\bar{F}(n,\xi^{(L)})=F(\check{n},\xi^{(L)})\,\beta_+^{-n_{\delta(+)}}\beta_-^{-n_{\delta(-)}}$$

where F was defined in Eq. (2.4); then

$$F(k,\xi^{(L)})=\bar{F}(\hat{k},\xi^{(L)})$$

Theorem 2.1. For every $t\in\mathbb{R}_+$

$$\int \bar{F}(\hat{k},\xi_t^{(L)})\,d\pi_{\xi_0^{(L)}}^{(L)}=\int \bar{F}(n_t,\xi_0^{(L)})\,d\mathbb{Q}_{\hat{k}}^{(L)}$$

where $\pi_{\xi_0}^{(L)}$ denotes the Markov process for the random variables $\xi_t^{(L)}$ generated by $G^{(L)}$ [see Eq. (2.3)] and starting at $\xi_{t=0}^{(L)}=\xi_0^{(L)}$.

Proof. This is the classical relation for association (see Ref. 5), and we only need to prove that

$$G^{(L)}\bar{F}(\hat{k},\xi^{(L)})=A_L\bar{F}(\hat{k},\xi^{(L)})$$

To prove this we will treat separately each term appearing in the sum defining $G^{(L)}$ [see Eq. (2.3)]. In the term "x" everything remains unchanged except $(\xi_x^{k_x}/k_x!)(\xi_{x+1}^{k_{x+1}}/k_{x+1}!)$; this is changed into

$$\begin{aligned}
E:=&\frac{1}{k_x!}\frac{1}{k_{x+1}!}\left[\int_0^1(\xi_x+\xi_{x+1})^{k_x+k_{x+1}}p^{k_x}(1-p)^{k_{x+1}}\,dp-\xi_x^{k_x}\xi_{x+1}^{k_{x+1}}\right]\\
=&\sum_{k=0}^{k_x+k_{x+1}}\xi_x^k\xi_{x+1}^{k_x+k_{x+1}-k}\frac{1}{k_x!\,k_{x+1}!}C_{k_x+k_{x+1}}^k\int_0^1p^{k_x}(1-p)^{k_{x+1}}\,dp\\
&-\frac{1}{k_x!\,k_{x+1}!}\xi_x^{k_x}\xi_{x+1}^{k_{x+1}}
\end{aligned}$$

where C_m^n, $n\leqslant m$, is the number of ways n elements can be chosen among m. Since

$$\int_0^1p^m(1-p)^n\,dp=\frac{m!\,n!}{(m+n)!\,(m+n+1)}$$

$$E=\frac{1}{k_x+k_{x+1}+1}\sum_{k=0}^{k_x+k_{x+1}}\left[\frac{\xi_x^k}{k!}\frac{\xi_{x+1}^{k_x+k_{x+1}-k}}{(k_x+k_{x+1}-k)!}-\frac{\xi_x^{k_x}}{k_x!}\frac{\xi_{x+1}^{k_{x+1}}}{k_{x+1}!}\right]$$

which is just the term corresponding to $A_L\bar{F}(\hat{k},\xi^{(L)})$ when in Eq. (2.7) i is chosen to be equal to x. For the boundary points, take L for instance, we have to replace $\xi_L^{k_L}/k_L!$ by [see again Eq. (2.3)]

$$\int_0^\infty \beta_+ e^{-\beta_+\xi} \frac{\xi^{k_L}}{k_L!}\, d\xi - \frac{\xi_L^{k_L}}{k_L!} = \beta_+^{-k_L} - \frac{\xi_L^{k_L}}{k_L!} \quad \blacksquare$$

Corollary 2.1. We have

$$\int F(k,\xi^{(L)})\, d\mu_L = \sum_{k_+ + k_- = |k|} \beta_+^{-k_+}\beta_-^{-k_-} q_L(k;k_+,k_-) \tag{2.8}$$

where μ_L is the only stationary measure for the process generated by $G^{(L)}$, and

$$q_L(k;k_+,k_-) = Q_k^L(\{k_+ \text{ particles will exit at } \delta_{(+)} \text{ and } k_- \text{ at } \delta_{(-)}\}) \tag{2.9}$$

Proof. It is obtained from Theorem 2.1 by letting t go to infinity so that on the left hand side we obtain the limiting distribution ξ_L for the process $\xi_t^{(L)}$. $\blacksquare$

3. ASYMPTOTIC BEHAVIOR OF q_L

The study of the stationary measure μ_L is reduced to that of q_L via Corollary 2.1 and so we will need probability estimates for the exit distribution of particles moving according to the process defined in Definition 2.1. It will be convenient to regard this process as imbedded in another one in which each particle has a label, this will be the x process defined in

Definition 3.1. x process. The particles have a label and move in I_L as follows: Choose x, $x+1$ $[-L \leqslant x \leqslant L-1]$ as in Definition 2.1 with equal probability. Particles sitting elsewhere than x, $x+1$ do not move. Compute the total number of those at x, $x+1$, let it be $n_x + n_{x+1}$. Choose the integer p uniformly between 0 and $n_x + n_{x+1}$ and independently a permutation of the $n_x + n_{x+1}$ labels of the particles at x and $x+1$. Then put the first p particles [i.e., those with the labels corresponding to the first p elements of the permutation] at x and the others at $x+1$. Like Definition 2.1 when $\delta(\pm)$ are chosen the particles at $\pm L$ transfer to $\delta(\pm)$ where they will remain forever.

It is clear from the definition that this process satisfies the properties of the n process, that is to say if we only observe the number of particles sitting at each site we recover the n process of Definition 2.1.

Finally denote by $p_L(x_1,\ldots,x_N;\epsilon_1,\ldots,\epsilon_N)$, $\epsilon_i = \pm 1$, $i = 1\ldots N$ the probability in the x process that particle i exists at $\delta(\epsilon_i)$ given the initial condition $x_1, \ldots, x_N$. We clearly have

$$q_L(n; k_+, k_-) = \sum{}^* p_L(x_1, \ldots, x_N; \epsilon_1, \ldots, \epsilon_N) \tag{3.1}$$

where

$$\sum_{i=1}^{N} 1_x(x_i) = n_x, \qquad 1_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

$$|n| = N$$

and $\sum^*$ is the sum over all the sequences ϵ_i, $i = 1 \ldots N$ such that the total number of 1's is k_+, i.e., $\sum_{i=1}^{N} \epsilon_i = 2k_+ - N$.

Proposition 3.1. Let $P^{(L)}_{x_1 \ldots x_n}$ be the x process with n particles initially at $x_1 \ldots x_n$. Let $m < n$, $i_1 \ldots i_m$ a subset of $1 \ldots n$, $y_j = x_{i_j}$, $j = 1 \ldots m$. Then the $p^{(L)}_{x_1 \ldots x_n}$ process for the random variables $x_{i_1}(t), \ldots, x_{i_m}(t)$ is isomorphic to the $p^{(L)}_{y_1 \ldots y_m}$ process of m particles with initial condition $y_1 \ldots y_m$. In particular for any $1 \leqslant i \leqslant n$

$$\sum_{\epsilon_i = \pm 1} p_L(x_1, \ldots, x_n; \epsilon_1, \ldots, \epsilon_n)$$
$$= p_L(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; \epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_{i+1}, \ldots, \epsilon_n) \tag{3.2}$$

Proof. Assume x, $x + 1$ are the sites involved in the displacement of particles at a given time. It is clear that the uniform distribution on the permutation of N particles induces the uniform distribution on the permutations of M of them. Therefore we only need to prove that the number of particles at x, after the rearrangement, has the correct law. This is equivalent to say that if we choose uniformly a subset of M numbers between 1 and N and then choose a number $0 \leqslant p \leqslant N$ then the number of elements of the subset that are smaller than p must have uniform law on $0, \ldots, M$. In other words take $X_1, \ldots, X_N$ independent random variables $P(X_i = 0) = P(X_i = 1) = 1/2$, then we want to prove that for every $0 \leqslant q \leqslant M$

$$\frac{1}{N+1} \sum_{p=0}^{N} P\left(\sum_{i=1}^{p} X_i = q \,\middle|\, \sum_{i=1}^{N} X_i = M \right) = \frac{1}{M+1} \tag{3.3}$$

We introduce the generating function for the variables

$$S_m = \sum_{i=1}^{m} X_i, \qquad 0 \leqslant m \leqslant N$$

$$T = \sum_{i=1}^{p} X_i$$

where p as before is distributed uniformly between 0 and N. Then we have for $x, y \in \mathbb{R}_+$

$$\mathbb{E}(x^{T}y^{S_N}) = \frac{1}{N+1}\sum_{p=0}^{N}\mathbb{E}(x^{S_p}y^{S_N}) = \frac{1}{N+1}\sum_{p=0}^{N}\mathbb{E}((xy)^{S_p}y^{S_N-S_p})$$
$$= \frac{1}{N+1}\frac{1}{2^N}\frac{(1+y)^{N+1}-(1+xy)^{N+1}}{y(1-x)} \tag{3.4}$$

On the other hand if we assume Eq. (3.3) to hold we have another way to compute the generating function and so we get

$$\mathbb{E}(x^{T}y^{S_N}) = \sum_{j=0}^{N}\frac{1}{2^N}C_N^j y^j(1+\cdots+x^j)\frac{1}{j+1}$$
$$= \frac{1}{y(1-x)}\frac{1}{(N+1)2^N}\left[(1+y)^{N+1}-(1+xy)^{N+1}\right]$$

which agrees with Eq. (3.4) and therefore the proposition is proven. ■

Proposition 3.2. Given any positive integer N, any $x_1 \ldots x_N$, $\epsilon_1 \ldots \epsilon_N$, any $u \in (-1, 1)$ and any permutation $\sigma(1) \ldots \sigma(N)$

$$\lim_{L\to\infty}\left[p_L(x_1+[uL], \ldots, x_N+[uL], \epsilon_1 \ldots \epsilon_N)\right.$$
$$\left.-p_L(x_1+[uL], \ldots, x_N+[uL], \epsilon_{\sigma(1)} \ldots \epsilon_{\sigma(N)})\right] = 0 \tag{3.5}$$

where $[uL]$ is the integer part of uL.

Proof. When two given particles are at neighboring sites they have a positive probability to exchange their names. By Proposition 3.1 we can use the same argument as in Lemma 3.5 of Ref. 3 and prove the proposition. ■

Proposition 3.3. Given any positive integer N, any $x_1 \ldots x_N$, $\epsilon_1 \ldots \epsilon_N$, any $u \in (-1, 1)$,

$$\lim\left[p_L(x_1+[uL], \ldots, x_N+[uL], \epsilon_1 \ldots \epsilon_N) - \prod_{i=1}^{N} p_L(x_i+[uL], \epsilon_i)\right] = 0$$

Proof. The proposition is proven to hold in the case $N = 2$ by using an analogous argument to that employed in Ref. 3 to prove Theorem 3.2 for the case $k = 2$. For every fixed u we define a measure on $\{-1, +1\}^{\mathbb{Z}\times N}$ by the following procedure. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be distinct elements in $\mathbb{Z} \times N$, $\eta(x_i, y_i)$ the corresponding variables taking values ± 1, then pose

$$\nu_L(\{\eta \mid \eta(x_i, y_i) = \epsilon_i, i = 1 \ldots n\})$$
$$= p_L(x_1+[uL], \ldots, x_n+[uL], \epsilon_1, \ldots, \epsilon_n)$$

By Proposition 3.1 ν_L is a probability measure and by Proposition 3.2 any weak limit is exchangeable. By using De Finetti's theorem together with the factorization property proven for $N = 2$ we obtain that any weak limit is Bernoulli; its parameter can be obtained using Proposition 3.1 and we have

$$\lim_{L\to\infty} \nu_L(\eta(x_1, y_1) = 1) = (1/2)(1+u) \tag{3.6}$$

which is the exit distribution for a single random walk. From this the proposition follows. ■

4. RESULTS

In Section 2 we have introduced the variables q_x, p_x, $\xi_x = q_x^2 + p_x^2$, $x \in \mathbb{Z}$, which denote, respectively, position momentum and energy of the oscillator at site x. We have then defined a stochastic evolution for the system of oscillators and we have proven the following.

Theorem 4.1. For every positive integer L the stochastic evolution defined in Section 2 for the oscillators in $-L, \ldots, L$ has a unique stationary measure $\tilde{\mu}_L$ where

$$d\tilde{\mu}_L = \left[\prod_{x=-L}^{L} \nu(dq_x\, dp_x \mid \xi_x)\right] d\mu_L(d\xi_{-L}, \ldots, d\xi_L) \tag{4.1}$$

where $\nu(dq_x\, dp_x \mid \xi_x)$ is the normalized Lebesgue (microcanonical) measure on the circle $q_x^2 + p_x^2 = \xi_x$ and μ_L is completely defined by Eq. (2.8).

Let O denote the algebra of cylindrical continuous bounded functions on $(\mathbb{R}\times\mathbb{R})^{\mathbb{Z}}$. Then we have the following.

Theorem 4.2. Let $\tau_{[uL]}$ denote the translation of the $[uL]$ = integer part of uL, $u \in (-1, 1)$; then

$$\lim_{L\to\infty} \tilde{\mu}_L(\tau_{[uL]} f) = \nu_{T(u)}(f), \qquad \forall f \in O \tag{4.2}$$

where

$$T(u) = T_-\left(\frac{1-u}{2}\right) + T_+\left(\frac{1+u}{2}\right) \tag{4.3}$$

and ν_T is the Gibbs measure for the (independent) oscillators at temperature T, namely,

$$d\nu_T = \prod_x \left[\nu(dq_x\, dp_x \mid \xi_x)\, \frac{1}{kT}\, e^{-(\xi_x/kT)}\, d\xi_x\right] \tag{4.4}$$

Proof. It is enough to study the convergence of the functions $F(k, \xi)$ [see Eq. (2.4)]. By Corollary 2.1, Eq. (2.8), we need to compute the limit of $q_L(k; k_+, k_-)$. By Eq. (3.1) this is related to the limit of $p_L(x_1 \ldots x_N, \epsilon_1, \ldots, \epsilon_N)$, which is obtained in Proposition 3.3 and Eq. (3.6). Therefore the theorem is proven. ■

We now want to compute the average energy flux between the oscillators at x, $x+1$, $-L \leqslant x \leqslant L-1$. We obviously have

$$Q_L := \int d\mu_L \int_0^1 \left[\xi_x - p(\xi_x + \xi_{x+1}) \right] dp = \int d\mu_L (1/2)(\xi_x - \xi_{x+1}) \quad (4.5)$$

Theorem 4.3. The heat flux Q_L defined in Eq. (4.5) is x independent:

$$Q_L = -\frac{1}{4L}(\beta_+^{-1} - \beta_-^{-1}) \quad (4.6)$$

$$Q = -\frac{K}{2}\frac{dT(u)}{du}, \qquad -1 < u < 1 \quad (4.7)$$

where $T(u)$ is defined by Eq. (4.3), K is Boltzmann's constant ($\beta^{-1} = KT$), and

$$Q = \lim_{L\to\infty} LQ_L \quad (4.8)$$

Equation (4.7) proves Fourier's law with heat conductivity coefficient $K/2$.

Proof. Equation (4.6) is a consequence of Eq. (4.5) and Corollary 2.1. The remainder is obvious. ■

Remark. The value of the heat conductivity depends linearly on the intensity of the process. In Section 2 we have chosen unit rate.

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REFERENCES

1. Z. Rieder, J. L. Lebowitz, and E. Lieb, *J. Math. Phys.* **8**:1073 (1967); H. Spohn and J. L. Lebowitz, *Commun. Math. Phys.* **54**:97 (1977).
2. M. Bolsterli, M. Rich, and W. M. Visscher, Simulation of non-harmonic interactions in a crystal by self-consistent reservoirs, *Phys. Rev. A* **1**:1086–1088 (1970); M. Rich and W. M. Visscher, Disordered harmonic chain with self consistent reservoirs, *Phys. Rev. B* **11**:2164–2170 (1975).
3. A. Galves, C. Kipnis, C. Marchioro, and E. Presutti, Non equilibrium measures which exhibit a temperature gradient: study of a model, *Commun. Math. Phys.* **81**:127–147 (1981).
4. V. B. Minassian, Some results concerning local equilibrium Gibbs distributions, *Sov. Math. Dokl.* **19**:1238–1242 (1978).
5. T. M. Liggett, The stochastic evolution of infinite systems of interacting particles, *Lecture Notes in Mathematics*, No. 598, pp. 188–248 (Springer, Berlin, 1976).
6. E. B. Davies, A model of heat conduction, *J. Stat. Phys.* **18**:161–170 (1978).